ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS AND MEROMORPHIC FUNCTIONS DEFINED BY THE LIU-SRIVASTAVA OPERATOR

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by

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CONTENTS

SYMBOLS

Symbol	Description
S	the class of all univalent function of the form $f(z) = z + a_2z^2 + \dots, \quad z \in \Delta$
$\mathcal{A}(p,m)$	the class of analytic functions $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n, \quad z \in \Delta$
$\mathcal{A}$	$\mathcal{A}(1,1)$
$T_g(p,m,\alpha)$	the class of p-valent functions in $\mathcal{A}(p,m)$ satisfying $\Re\left(\frac{(f*g)(z)}{z^p}\right) > \alpha$
$T_g[p,m,\alpha]$	the class of p-valent analytic functions in $T_g(p,m,\alpha)$ with negative coefficients.
P	the class of functions p analytic in Δ with positive real part and $p(0) = 1$
Δ	open unit disk $\{z : \lvert z\rvert < 1\}$
Δ^*	punctured unit disk $\Delta - \{0\}$
Δ_r	open disk of radius $r \quad \{z : \lvert z\rvert < r\}$
Σ_p	the class of all functions of the form $f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k, z \in \Delta^*$
$\mathcal{N}$	set of positive integers
$(a)_n$	Pochhammer symbol
$H_p^{l,m}$	Liu-Srivastava linear operator
$\prec$	subordinate to

arg	argument
$k(z)$	Koebe function
$I_p(n, \lambda)$	multiplier transform
S^*	Starlike functions in S
C	Convex functions in S
T	class of all analytic functions in $\mathcal{A}$ of the form
	$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0)$
$TS^*(\alpha)$	subclass of T consisting of starlike functions of order α
$TC(\alpha)$	subclass of T consisting of convex functions of order α
$f * g$	convolution or Hadamard product of functions f and g
$\mathcal{C}$	Complex plane
Σ	the class of univalent and analytic functions in Δ^*
$\Re$	Real part of a complex number
$\Im$	Imaginary part of a complex number

ABSTRACT

The present work is devoted to the study of certain subclasses of analytic functions and meromorphic functions defined in the unit disk $\Delta = \{z : |z| < 1\}$.

Let $\mathcal{A}(p, m)$ consists of analytic p-valent functions of the form $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n$ in Δ. A subclass of $\mathcal{A}(p, m)$ with negative coefficients is introduced. We obtain coefficient inequalities for this subclass. Distortion and growth estimates for functions in this class as well as inclusion and closure properties are also determined. A representation theorem is derived and it is proved that the subclass is closed under the Bernardi integral operator.

Let Σ_p be the class of meromorphic functions of the form $f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k$ defined in the unit disk Δ. Functions in Σ_p are analytic in the punctured unit disk $\Delta^* = \Delta - \{0\}$. Inequalities are obtained for meromorphic functions in Σ_p which are associated with the Liu-Srivastava linear operator $H_p^{l,m}$ and the multiplier transform $I_p(n, \lambda)$. In addition, we obtain sufficient conditions for $f \in \Sigma_p$ to satisfy a growth inequality.

FUNGSI ANALISIS BERPEKALI NEGATIF DAN FUNGSI MEROMORFIK YANG DIJANAKAN MELALUI PENJELMAAN LIU-SRIVASTAVA

ABSTRAK

Kajian ditumpukan kepada beberapa subkelas fungsi analisis dan fungsi meromorfik yang tertakrif pada cakera unit $\Delta = \{z : |z| < 1\}$.

Andaikan $\mathcal{A}(p, m)$ mengandungi fungsi-fungsi analisis p-valen yang berbentuk $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n$ pada Δ. Suatu subkelas bagi $\mathcal{A}(p, m)$ dengan pekali negatif diperkenalkan. Kita mendapatkan batas atas pekali-pekali bagi subkelas tersebut. Anggaran perubahan dan pertumbuhan bagi fungsi dalam kelas ini diperolehi, disamping sifat rangkuman dan tertutup. Teorem perwakilan bagi fungsi dalam kelas ini dibuktikan dan penjelmaan kamiran Bernardi ke atas kelas ini dikaji.

Andaikan Σ_p ialah kelas fungsi meromorfik berbentuk $f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k$ yang tertakrif pada cakera unit . Fungsi dalam Σ_p adalah analisis di dalam cakera unit berlubang $\Delta^* = \Delta - \{0\}$. Ketaksamaan yang melibatkan fungsi-fungsi dalam Σ_p yang dijanakan melalui penjelmaan linear Liu-Srivastava $H_p^{l,m}$ dan penjelmaan kepelbagaian $I_p(n, \lambda)$ diperoleh. Kita juga turut memperolehi syarat cukup bagi $f \in \Sigma_p$ untuk memenuhi suatu ketaksamaan pertumbuhan.

CHAPTER 1

INTRODUCTION

1.1. Univalent functions

Let $\Delta = \{z \in \mathcal{C} : |z| < 1\}$ be the open unit disk in the complex plane. Let $\mathcal{A}$ consists of all analytic functions $f : \Delta \to \mathcal{C}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. A function $f \in \mathcal{A}$ is univalent in Δ if it is one-to-one in Δ. Let S denote the class consisting of all analytic univalent functions $f(z)$ in $\mathcal{A}$. Thus functions in S have the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \quad z \in \Delta.$$

Two important functions in S are given by the following examples.

Example 1. The Koebe function is defined by

$$k(z) = \frac{z}{(1-z)^2}.$$

If $k(z_1) = k(z_2)$, then $z_1 - 2z_1 z_2 + z_1 z_2^2 = z_2 - 2z_1 z_2 + z_1^2 z_2$ which implies $(z_1 - z_2)(1 - z_1 z_2) = 0$.

Since $z_1, z_2 \in \Delta$ implies $|z_1 z_2| < 1$, we have $1 - z_1 z_2 \neq 0$. Hence $z_1 = z_2$. This shows that $k(z)$ is univalent in Δ.

Clearly $k(0) = 0 = k'(0) - 1$. Thus $k(z) = \frac{z}{(1-z)^2} \in S$. The function k maps Δ univalently onto $\mathcal{C} \setminus (-\infty, -1/4]$. The Koebe function is an extremal function for many problems in the class S.

Example 2. The function $\tau(z) = \frac{z}{1-z}$ belongs to S. This can be shown in a similar manner as in the previous example.

1.2. The Bierberbach conjecture

In 1916, Bieberbach proved that if $f(z) = z + a_2 z^2 + \ldots \in S$, then $|a_2| \leq 2$ and that equality holds if and only if $f(z) = k(z) = \frac{z}{(1-z)^2}$ or one of its rotations. He also conjectured that

$$|a_n| \leq n$$

for $n \geq 2$ and that equality holds if and only if $f(z)$ is either the function $k(z) = \frac{z}{(1-z)^2}$ or one of its rotations. It was only in 1985 that de Branges [**4**] provided a complete solution to the Bierberbach conjecture. Prior to this achievement, the Bierberbach conjecture was shown to be true for several subclasses of univalent functions. These classes include the class of starlike functions, the class of convex functions and the class of close-to convex functions. We shall define these classes later. We now give a proof of Bieberbach's result.

Theorem 1.2.1 (Bieberbach). *Let $f(z) = z + a_2 z^2 + \ldots \in S$. Then $|a_2| \leq 2$ and equality holds if and only if $f(z) = \frac{z}{(1-z)^2}$ or its rotations.*

To prove this theorem we need the following lemma.

Lemma 1.2.2. [**12**] *If* $w(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$ *is univalent in* $|z| > 1$, *then*

$$\sum_{n=1}^{\infty} n|a_n|^2 \leq 1.$$

Proof of Theorem 1.2.1. We first show that the function $F(z) = \sqrt{f(z^2)} = z + \frac{a_2 z^3}{2} + \ldots$ is univalent in $|z| < 1$. If $F(z_1) = F(z_2)$ then $f(z_1^2) = f(z_2^2)$. However $f(z)$ being univalent yields $z_1^2 = z_2^2$, that is, $z_1 = \pm z_2$. Since $F(z)$ is an odd function, $z_1 = -z_2$ implies that $F(z_1) = -F(z_2)$. Thus the only solution of $F(z_1) = F(z_2)$ is $z_1 = z_2$ and consequently $F(z)$ is univalent. It follows that

$$\phi(z) = [F(1/z)]^{-1} = z - \frac{a_2}{2z} + \frac{c_3}{z^3} + \frac{c_5}{z^5} + \ldots \tag{1.2.1}$$

is an odd univalent function in $|z| > 1$. Hence, by Lemma 1.2.2

$$\frac{1}{4}|a_2|^2 + 3|c_3|^2 + \ldots \leq 1.$$

Then

$$|a_2|^2 \leq 4 \Rightarrow |a_2| \leq 2.$$

If $|a_2| = 2$ then $a_2 = 2e^{i\alpha}$ so that $\left|\frac{-1}{2}a_2\right|^2 = 1$. Since the coefficients of the function $\phi(z)$ in (1.2.1) satisfies

$$\left|\frac{-1}{2}a_2\right|^2 + 0 + 3|c_3|^2 + 0 + 5|c_5|^2 + \ldots \leq 1,$$

$|a_2| = 2$ implies

$$1 + 3|c_3|^2 + 0 + 5|c_5|^2 + \ldots \leq 1.$$

Consequently, $c_{2n+1} = 0, \quad \forall n \geq 1.$

The function $\phi(z)$ in (1.2.1) takes the form of

$$\phi(z) = \frac{1}{F(1/z)} = z - \frac{e^{i\alpha}}{z}, \quad i.e, \quad F(z) = \frac{z}{1 - e^{i\alpha} z^2}.$$

Thus

$$f(z^2) = [F(z)]^2 = \frac{z^2}{(1 - e^{i\alpha} z^2)^2}.$$

In other words, $f(z) = \frac{z}{(1-e^{i\alpha}z)^2}$, which is a rotation of the Koebe function. $\square$

By using the bound for $|a_2|$, we now obtain bounds for $|f'(z)|$ and $|f(z)|$. In the proof of these results, we need the following lemmas.

LEMMA 1.2.3. *If $f(z)$ is in S, then for any ς in Δ,*

$$\frac{1}{2}\left|\frac{f''(\varsigma)}{f'(\varsigma)}(1 - |\varsigma|^2) - 2\overline{\varsigma}\right| \leq 2.$$

PROOF. For each fixed ς in Δ, the function

$$g(z) = \frac{f(\frac{z+\varsigma}{1+\varsigma z}) - f(\varsigma)}{f'(\varsigma)(1 - |\varsigma|^2)} = z + b_2 z^2 + \dots$$

is in S. By Theorem 1.2.1, we have

$$|b_2| = \left|\frac{f''(\varsigma)(1 - |\varsigma|^2)}{2f'(\varsigma)} - \overline{\varsigma}\right| \leq 2$$

which yields the desired result. $\square$

LEMMA 1.2.4. *If $f(z)$ is analytic at $\varsigma = \varrho e^{i\theta}$ and $f'(\varsigma) \neq 0$, then*

$$\varrho \frac{\partial}{\partial \varrho} \ln |f'(\varsigma)| = \Re \varsigma \frac{f''(\varsigma)}{f'(\varsigma)}$$

and

$$\varrho \frac{\partial}{\partial \varrho} \arg(f'(\varsigma)) = \Im \varsigma \frac{f''(\varsigma)}{f'(\varsigma)}.$$

Proof. Since $\log f'(\varsigma) = \ln|f'(\varsigma)| + i\arg f'(\varsigma)$, we have

$$\varrho\frac{\partial}{\partial\varrho}\log f'(\varsigma) = \varrho\frac{\partial}{\partial\varrho}(\ln|f'(\varsigma)| + i\arg(f'(\varsigma)).$$

Also

$$\begin{aligned}\varrho\frac{\partial}{\partial\varrho}\log f'(\varsigma) = \varrho\frac{\partial}{\partial\varsigma}(\log f'(\varsigma))\frac{\partial\varsigma}{\partial\varrho} &= \varrho\frac{f''(\varsigma)}{f'(\varsigma)}e^{i\theta} \\ &= \varsigma\frac{f''(\varsigma)}{f'(\varsigma)}.\end{aligned}$$

Therefore

$$\varsigma\frac{f''(\varsigma)}{f'(\varsigma)} = \varrho\frac{\partial}{\partial\varrho}(\ln|f'(\varsigma)|) + i\varrho\frac{\partial}{\partial\varrho}(\arg f'(\varsigma)).$$

The result is obtained by comparing the real and imaginary part. □

1.3. Distortion and growth theorems

Theorem 1.3.1 (Distortion theorem). *If $f(z)$ is in S, then for each $z = re^{i\theta}$ in Δ,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}.$$

These inequalities are sharp. Equality occurs for the Koebe function

$$k(z) = \frac{z}{(1-z)^2}.$$

Proof. Multiplying the inequality in Lemma 1.2.3 by $\frac{2|\varsigma|}{1-|\varsigma|^2}$ yields

$$\left|\frac{\varsigma f''(\varsigma)}{f'(\varsigma)} - \frac{2|\varsigma|^2}{1-|\varsigma|^2}\right| \leq \frac{4|\varsigma|}{1-|\varsigma|^2}.$$

Writing $|\varsigma|$ by ϱ and using the fact that

$$|\alpha| \leq \beta \Rightarrow -\beta \leq \Re\alpha \leq \beta, \quad -\beta \leq \Im\alpha \leq \beta,$$

we get

$$\frac{-4\varrho}{1-\varrho^2} \leq \Re\left(\frac{\varsigma f''(\varsigma)}{f'(\varsigma)} - \frac{2\varrho^2}{1-\varrho^2}\right) \leq \frac{4\varrho}{1-\varrho^2}$$

and

$$\frac{-4\varrho}{1-\varrho^2} \leq \Im\left(\frac{\varsigma f''(\varsigma)}{f'(\varsigma)} - \frac{2\varrho^2}{1-\varrho^2}\right) \leq \frac{4\varrho}{1-\varrho^2}.$$

We transpose the real term $\frac{2\varrho^2}{1-\varrho^2}$ and use Lemma 1.2.4 to get

$$\frac{2\varrho^2-4\varrho}{1-\varrho^2} \leq \varrho\frac{\partial}{\partial\varrho}\ln|f'(\varsigma)| \leq \frac{2\varrho^2+4\varrho}{1-\varrho^2}$$

and

$$\frac{4\varrho}{1-\varrho^2} \leq \varrho\frac{\partial}{\partial\varrho}\arg f'(\varsigma) \leq \frac{4\varrho}{1-\varrho^2}.$$

We divide by ϱ the first of the above two inequalities and integrate along the straight line path from $z=0$ to $z=re^{i\theta}$ (ϱ runs from 0 to r and $f'(0)=1$) to obtain

$$\ln\frac{1-r}{(1+r)^3} \leq \ln|f'(z)| \leq \ln\frac{1+r}{(1-r)^3}.$$

We get the desired result by exponentiating. Since for the Koebe function

$$k'(z) = \frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) = \frac{1+z}{(1-z)^3},$$

the inequalities are sharp at $z=r$ and $z=-r$. □

We need the following lemma to prove the growth theorem for univalent function.

LEMMA 1.3.2 (10, Theorem 7, p.67). *Suppose that $f(z)\in S$ satisfies*

$$m'(r) \leq |f'(z)| \leq M'(r), \quad (0 \leq r < 1, |z| \leq r)$$

where $m'(r)$ and $M'(r)$ are real-valued functions of r in $[0,1)$. Then

$$\int_0^r m'(t)dt \leq |f(z)| \leq \int_0^r M'(t)dt.$$

THEOREM 1.3.3 (Growth Theorem). *If $f(z) \in S$, then for $|z| \leq r$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}.$$

Equality occurs for

$$k(z) = \frac{z}{(1-z)^2}.$$

PROOF. The result follows by taking

$$m'(r) = \frac{1-r}{(1+r)^3} \text{ and } M'(r) = \frac{1+r}{(1-r)^3}$$

in Lemma 1.3.2. □

Since

$$\lim_{r\to 1^+} \frac{r}{(1+r)^2} = \frac{1}{4},$$

the image of the unit disk under the mapping $w = f(z)$ contains the disk of radius $1/4$. This disk is called the Koebe domain.

THEOREM 1.3.4 (Koebe domain). *If $f \in S$, then*

$$f(\Delta) \supseteq \{w : |w| \leq 1/4\}.$$

The result is sharp for $k(z) = \frac{z}{(1-z)^2}$.

1.4. Convex and starlike functions

Definition 1.4.1. A set D in the complex plane is said to be *starlike with respect* to an interior point w_0 in D if for each $w \in D$, the line segment joining w and w_0 lies entirely in D, i.e., $tw_0 + (1-t)w \in D$ for $0 \leq t \leq 1$. If a function $f \in S$ maps Δ onto a domain that is *starlike with respect* to w_0, then we say that $f(z)$ is *starlike with respect* to w_0. In the special case that $w_0 = 0$, we simply say that $f(z)$ is a *starlike* function.

The class of *starlike* functions in S will be denoted by S^*. The Koebe function, $k(z) = \frac{z}{(1-z)^2}$ is an extremal function for many problems in the class S^*.

Definition 1.4.2. A set D in the complex plane is called *convex* if for every pair of points w_1 and w_2 in D, the line segment joining w_1 and w_2 lies also in D. If a function $f \in S$ maps Δ onto a *convex* domain, then $f(z)$ is called a *convex* function.

Note that D is *convex* if it is starlike with respect to every point $w_0 \in D$. Further, for every pair of points w_1 and w_2 in D, the point $tw_1 + (1-t)w_2 \in D$ for $0 \leq t \leq 1$.

The class of convex functions in S will be denoted by C. The function $\tau(z) = \frac{z}{1-z}$ is an extremal function for many problems in the class C.

1.5. Convexity and starlikeness of a curve

Consider the image of a curve Γ_z under a function $f(z)$ that is analytic on Γ_z. In most cases, Γ_z will either be a circle, a line segment, or some other elementary arc. Hence we shall assume that Γ_z is a smooth curve with parametrization

$$z(t) = x(t) + iy(t) \quad (a \leq t \leq b)$$

where $x(t)$ and $y(t)$ are real functions, and

$$z'(t) = x'(t) + iy'(t) \neq 0$$

for t in $[a, b]$. The arc Γ_z is a directed arc, the direction being that determined as t increases. Let Γ_w be the image of Γ_z under a function $f(z)$ that is analytic on Γ_z and assume that w_0 is not on Γ_w. The arc Γ_w is said to be starlike with respect to w_0 if $\arg(w - w_0)$ is a nondecreasing function of t, i.e, if

$$\frac{d}{dt}(\arg(w - w_0)) \geq 0$$

$t \in [a, b]$. To convert this inequality to a more useful form, we have

$$\begin{aligned}
\frac{d}{dt} \arg(w - w_0) &= \frac{d}{dt} \Im \log(w - w_0) \\
&= \Im \left[\frac{d}{dt} \log(w - w_0) \right] \\
&= \Im \left[\frac{d}{dz} \log(w - w_0) \frac{dz}{dt} \right] \\
&= \Im \left[\frac{f'(z)}{f(z) - w_0} \frac{dz}{dt} \right].
\end{aligned}$$

Lemma 1.5.1. *The image of $\Gamma_z : z = z(t)$ under $f(z)$ is starlike with respect to w_0 if and only if*

$$\Im\left[\frac{f'(z)}{f(z)-w_0}z'(t)\right] \geq 0, \quad t \in [a,b]. \tag{1.5.1}$$

The arc Γ_w is said to be convex if the argument of the tangent to Γ_w is a nondecreasing function of t. The direction of the tangent to Γ_z is argument $z'(t)$ and the mapping $w = f(z)$ rotates this tangent vector through an angle of argument $f'(z)$. Thus the arc Γ_w is a convex arc if and only if

$$\frac{dT}{dt} = \frac{d}{dt}\left[\arg(z'(t)f'(z))\right] \geq 0, \quad t \in [a,b].$$

The same technique used to derive equation (1.5.1) gives

$$\begin{aligned}
\frac{d}{dt}\arg(z'(t)f'(z)) &= \frac{d}{dt}\Im\left[\log z'(t) + \log f'(z)\right] \\
&= \Im\left[\frac{z''(t}{z'(t)} + \frac{d}{dz}\log f'(z)\frac{dz}{dt}\right] \\
&= \Im\left[\frac{z''(t)}{z'(t)} + \frac{f''(z)}{f'(z)}z'(t)\right].
\end{aligned}$$

Lemma 1.5.2. *Suppose that $f'(z) \neq 0$ on $\Gamma_z : z = z(t)$. Then the image of Γ_z under $f(z)$ is a convex arc if and only if*

$$\Im\left[\frac{z''(t)}{z'(t)} + \frac{f''(z)}{f'(z)}z'(t)\right] \geq 0, \quad t \in [a,b]. \tag{1.5.2}$$

We now specialize these formulas by selecting Γ_z to be the circle $C_R : |z| = R$ with the usual orientation $z = Re^{it}, \quad 0 \leq t \leq 2\pi$. In this case, $z'(t) = Re^{it} = iz$ and $z''(t) = -Re^{it} = -z$. The inequality (1.5.1) becomes

$$\Im\left[\frac{izf'(z)}{f(z)-w_0}\right]=\Re\left[\frac{zf'(z)}{f(z)-w_0}\right]\geq 0,$$

while inequality (1.5.2) becomes

$$\Im\left[i+\frac{izf''(z)}{f'(z)}\right]=\Re\left[1+\frac{zf''(z)}{f'(z)}\right]\geq 0$$

for z on C_R. Thus, a normalized analytic function f is *starlike* if and only if f satisfies $\Re\left\{\frac{zf'(z)}{f(z)}\right\}>0$, while f is *convex* if and only if $\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0$. In general, we say that $f\in S$ is *starlike* of order α, if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\quad (0\leq\alpha<1)$$

and *convex* of order α if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha,\quad (0\leq\alpha<1).$$

The classes of such functions are denoted by $S^*(\alpha)$ and $C(\alpha)$.

THEOREM 1.5.3. *If $f(z)$ is in $S^*(\alpha)$, then for $|z|\leq r$,*

$$\frac{r}{(1+r)^{2(1-\alpha)}}\leq|f(z)|\leq\frac{r}{(1-r)^{2(1-\alpha)}}$$

$$\frac{r}{(1+r)^{3-2\alpha}}\leq|f'(z)|\leq\frac{r}{(1-r)^{3-2\alpha}}$$

All inequalities are sharp, with equality if and only if $f(z)$ is a rotation of $k(z,\alpha)=\frac{z}{(1-z)^{2(1-\alpha)}}$.

THEOREM 1.5.4. *Let $f(z)$ be in $C(\alpha)$. Then for $|z|\leq r$,*

$$\frac{1}{(1+r)^{2(1-\alpha)}}\leq|f'(z)|\leq\frac{1}{(1-r)^{2(1-\alpha)}}.$$

If $\alpha \neq 1/2$*, then*

$$\frac{(1+r)^{2\alpha-1}-1}{2\alpha-1} \leq |f(z)| \leq \frac{1-(1-r)^{2\alpha-1}}{2\alpha-1}$$

and if $\alpha = 1/2$*, then*

$$\ln(1+r) \leq |f(z)| \leq -\ln(1-r).$$

All inequalities are sharp. The extremal functions are rotations of

$$f(z) = \frac{1-(1-z)^{2\alpha-1}}{2\alpha-1}$$

if $\alpha \neq 1/2$*, and*

$$f(z) = -\log(1-z)$$

for $\alpha = 1/2$*.*

THEOREM 1.5.5 (Alexander Theorem). *Let* $f \in \mathcal{A}$*. Then* $f(z) \in C(\alpha)$ *if and only if* $F(z) = zf'(z) \in S^*(\alpha)$

PROOF. Since $F(z) = zf'(z)$, we have

$$\frac{zF'(z)}{F(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

Hence

$$\Re\frac{zF'(z)}{F(z)} > \alpha$$

if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha.$$

Thus $f \in C(\alpha)$ if and only if $zf' \in S^*(\alpha)$. $\square$

1.6. Univalent functions with negative coefficients

Let $T \subset S$ be the class of all analytic univalent functions $f(z)$ with negative coefficients of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

For $0 \leq \alpha < 1$, let $TS^*(\alpha)$ and $TC(\alpha)$ be the subclasses of T consisting of starlike functions of order α and convex functions of order α respectively. Thus $TS^*(\alpha) = T \cap S^*(\alpha)$ and $TC(\alpha) = T \cap C(\alpha)$. These classes of functions with negative coefficients were introduced and studied by Silverman [**23**]. The following results are known.

THEOREM 1.6.1. [**23**] *A function* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in TS^*(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} (n - \alpha) a_n \leq 1 - \alpha.$$

COROLLARY 1.6.2. [**23**] *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in TS^*(\alpha)$, *then*

$$a_n \leq \frac{1 - \alpha}{n - \alpha}$$

and the result is sharp for $f(z) = z - \frac{1-\alpha}{n-\alpha} z^n$.

THEOREM 1.6.3. [**23**] *If* $f \in TS^*(\alpha)$, *then for* $|z| \leq r$,

(1) $r - \frac{1-\alpha}{2-\alpha} r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{2-\alpha} r^2$ *with equality for*

$$f(z) = z - \frac{(1 - \alpha)}{2 - \alpha} z^2.$$

(2) $1 - \frac{2(1-\alpha)}{2-\alpha} r \leq |f'(z)| \leq 1 + \frac{2(1-\alpha)}{2-\alpha} r$ *and the result is sharp for*

$$f(z) = z - \frac{1 - \alpha}{2 - \alpha} z^2 \quad (z = \pm r).$$

Theorem 1.6.4. [**23**] *A function* $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$ *is in* $TC(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}n(n-\alpha)a_n\leq 1-\alpha.$$

Corollary 1.6.5. [**23**] *If* $f(z)=z-\sum_{n=2}^{\infty}a_nz^n\in TC(\alpha)$, *then*

$$a_n\leq\frac{1-\alpha}{n(n-\alpha)}$$

and the result is sharp for $f(z)=z-\frac{(1-\alpha)}{n(n-\alpha)}z^n$.

Theorem 1.6.6. [**23**] *If* $f\in TC(\alpha)$, *then for* $|z|\leq r$,

(1) $r-\frac{1-\alpha}{2(2-\alpha)}r^2\leq|f(z)|\leq r+\frac{1-\alpha}{2(2-\alpha)}r^2$ *and the result is sharp for*

$$f(z)=z-\frac{(1-\alpha)z^2}{2(2-\alpha)}\quad(z=\pm r).$$

(2) $1-\frac{1-\alpha}{2-\alpha}r\leq|f'(z)|\leq 1+\frac{1-\alpha}{2-\alpha}r$ *and the result is sharp for*

$$f(z)=z-\frac{(1-\alpha)z^2}{2(2-\alpha)}\quad(z=\pm r).$$

1.7. Functions with positive real part

A function $p(z)=1+c_1z+\ldots$ is called a function with positive real part provided

$$\Re p(z)>0,\quad(z\in\Delta).$$

The class of all such function is denoted by P. More generally we denote by $P(\alpha)$ the class of analytic functions, $p\in P$ with

$$\Re p(z)>\alpha\quad(0\leq\alpha<1).$$

Theorem 1.7.1. [**10**] *If $p(z) \in P$, then for each fixed z in Δ with $|z| \leq r$, $p(z)$ lies in the closed disk with center at $(1+r^2)/(1-r^2)$ and radius $2r/(1-r^2)$, i.e.,*

$$\left|p(z) - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}.$$

The diameter end points of the disk that contains $p(z)$ are $(1-r)/(1+r)$ and $(1+r)/(1-r)$.

To prove the theorem we use the Lindelof principle which is really a natural extension of Schwarz Lemma.

Definition 1.7.2. Let B_0 denote the set of all functions of the form

$$b(z) = \sum_{n=1}^{\infty} b_n z^n = b_1 z + \dots$$

that are analytic in Δ and for which $|b(z)| < 1$ in Δ. Thus $b(z)$ is analytic in Δ, $b(0) = 0$ and $|b(z)| \leq 1$ for z in Δ. Such a function $b(z)$ is called a Schwarzian function.

Theorem 1.7.3 (Schwarz Lemma). *Let $b(z)$ be in B_0. Then for each r with $0 \leq r < 1$,*

$$\left|b(re^{i\theta})\right| \leq r \tag{1.7.1}$$

If equality occurs in equation (1.7.1) at one point $z_0 = re^{i\theta}$ with $0 \leq r < 1$, then $b(z) = e^{i\alpha} z$ for some real α. Additionally

$$|b_1| = |b'(0)| \leq 1$$

and $|b_1| = 1$ if and only if $b(z) = B(z) = e^{i\alpha} z$.

Theorem 1.7.3 tells us that the univalent function $B(z) = e^{i\alpha}z$ is in some sense a maximal function among all bounded functions with $b(0) = 0$, or the bounded functions $b(z)$ are "subordinate" to the univalent function $B(z)$. We make this concept of subordination precise, and we extend it to an arbitrary function in the following definition.

Definition 1.7.4 (Subordination). Let $F(z) = a_0 + a_1 z + \ldots$ be analytic and univalent in Δ and suppose that $F(\Delta) = D$. If $f(z)$ is analytic in Δ, $f(0) = F(0)$, and $f(\Delta) \subset D$, the we say that $f(z)$ is subordinate to $F(z)$ in Δ, and we write

$$f(z) \prec F(z).$$

We also say that $F(z)$ is superordinate to $f(z)$ in Δ.

We observe that in this definition $F(z)$ is univalent in Δ, but nothing is assumed about the valence of $f(z)$. Both $F(z)$ and $f(z)$ carry $z = 0$ into the same point, and $f(z)$ carries Δ onto some (possibly multi-sheeted) surface whose projection onto the plane is contained in D. For example, under the conditions on $b(z)$ in Schwarz Lemma, $b(z) \prec B(z) = e^{i\alpha}z$. Suppose now that $f(z) \prec F(z)$ and $F(\Delta) = D$. Then the inverse F^{-1} is analytic in D and maps D onto Δ with $F^{-1}(a_0) = 0$. Hence the composite function $b(z) = F^{-1}(f(z))$ is analytic in Δ, and maps Δ into Δ. Further $b(0) = F^{-1}(f(0)) = F^{-1}(a_0) = 0$. Thus $b(z)$ is a Schwarzian function, and $f(z) = F(b(z))$. We have proved the "only if" part of the following theorem.

THEOREM 1.7.5. *Let $f(z)$ and $F(z)$ be analytic in Δ, and suppose that $F(z)$ is univalent in Δ. Then $f(z) \prec F(z)$ in Δ if and only if there exists a Schwarzian function $b(z)$ that satisfies*

$$f(z) = F(b(z)). \tag{1.7.2}$$

The proof of the "if" part of this theorem is trivial. We note that in the definition of subordination we assume that $F(z)$ is univalent in Δ. The concept of subordination can be extended to the case where $F(z)$ is not univalent, and the simplest way to do this is to use equation (1.7.2) as its definition. Thus, $f(z) \prec F(z)$ if and only if there is a $b(z)$ in B_0 such that $f(z) = F(b(z))$. As an example, if n is a positive integer, then $z^n \prec z$ in Δ. If we do not demand that $F(z)$ is univalent, then $z^{2n} \prec z^2$ in Δ, but z^{2n+1} is not subordinate to z^2 in Δ.

THEOREM 1.7.6 (Lindelof Principle). *Suppose that $f(z) \prec F(z)$ in Δ. Then for each r in $[0,1]$*

$$f(\Delta_r) \subset F(\Delta_r)$$

where $\Delta_r = \{z : |z| \leq r\}$. Further, if $f(re^{i\theta})$ is on the boundary of $F(\Delta_r)$ for one point $z_0 = re^{i\theta}$ with $0 < r < 1$, then there is a real α such that $f(z) = F(e^{i\alpha}z)$ and $f(re^{i\theta})$ is on the boundary of $F(\Delta_r)$ for every point $z = re^{i\theta}$ in Δ.

PROOF. If $f(z) \prec F(z)$ then by equation (1.7.2) there is a $b(z)$ such that $f(z) = F(b(z))$ and $b(z)$ satisfies the conditions of Schwarz's

Lemma. Then $f(\Delta_r) = F(b(\Delta_r)) \subset F(\Delta_r)$. If there is a z_0 such that $0 < r = |z_0| < 1$ and $f(z_0)$ is a boundary point of $F(\Delta_r)$, then $|b(z_0)| = r$. Hence $b(z) = e^{i\alpha}z$ for some real α and $f(z) = F(e^{i\alpha}z)$. $\square$

1.8. The Herglotz representation formula

THEOREM 1.8.1 (Herglotz). *If $p(z) \in P$, then there is a real-valued nondecreasing function $\mu(\phi)$ such that*

$$\int_0^{2\pi} d\mu(\phi) = 2\pi \tag{1.8.1}$$

and for each z in Δ

$$p(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{-i\phi}}{1-ze^{-i\phi}}d\mu(\phi) = \frac{1}{2\pi}\int_0^{2\pi} L_0(e^{-i\phi z})d\mu(\phi). \tag{1.8.2}$$

Conversely, if $p(z)$ is defined by equation(1.8.2) and $\mu(\phi)$ is a nondecreasing function satisfying equation (1.8.1), then $p(z) \in P$.

COROLLARY 1.8.2. *If*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z_n \tag{1.8.3}$$

is in P, then for all $n \geq 1$

$$p_n = \frac{1}{\pi}\int_0^{2\pi} e^{-in\phi}d\mu(\phi).$$

THEOREM 1.8.3. *Let $p(z)$ given by (1.8.3) be analytic in Δ and let $U(z)$ and $V(z)$ denote the real and imaginary parts of $p(z)$ respectively. Set $z = re^{i\theta}$ and $\varsigma = \varrho e^{i\phi}$. If $|z| < |\varsigma| < 1$, then*

$$p(z) = \frac{1}{2\pi}\int_0^{2\pi} U(\varrho e^{i\theta})\frac{\varsigma + z}{\varsigma - z}d\phi + iV(0). \tag{1.8.4}$$

This formula for $p(z)$ is often referred to as the Schwarz's representation formula. It expresses $p(z)$ in terms of its real part on a slightly larger circle $|\varsigma| = \varrho > |z|$.

Proof. Using Cauchy's formula and integrating on the circle $|\varsigma| = \varrho < 1$ we have

$$p_n = \frac{1}{2\pi i}\int_{|\varsigma|=\rho} \frac{p(\varsigma)}{\varsigma^{n+1}} d\varsigma = \frac{1}{2\pi \varrho^n}\int_0^{2\pi} p(\varsigma) e^{-in\phi} d\phi \tag{1.8.5}$$

for $n = 1, 2, \ldots$. If we put ς^{n-1} in the numerator, we have instead

$$0 = \frac{1}{2\pi i}\int_{|\varsigma|=\rho} p(\varsigma)\varsigma^{n-1} d\varsigma = \frac{\varrho^n}{2\pi}\int_0^{2\pi} p(\varsigma) e^{in\phi} d\phi \tag{1.8.6}$$

for $n = 1, 2, \ldots$. We divide the above equation by ϱ^{2n}, take the conjugate and add the result to equation (1.8.5). This gives

$$\begin{aligned} p_n &= \frac{1}{2\pi \varrho^n}\int_0^{2\pi} (p(\varsigma) + \overline{p(\varsigma)}) e^{-in\phi} d\phi \\ &= \frac{1}{2\pi \varrho^n}\int_0^{2\pi} 2U(\varsigma) e^{-in\phi} d\phi \end{aligned}$$

for $n = 1, 2, \ldots$ For each such n, we multiply the second of the above equation by z^n and add the resulting equations. Since $|z| < \varrho$, the convergence of the infinite series we obtain is assured. Thus

$$\begin{aligned} p(z) = p_0 + \sum_{n=1}^{\infty} p_n z^n &= \frac{1}{2\pi}\int_0^{2\pi} p(\varsigma) d\phi + \sum_{n=1}^{\infty} \frac{z^n}{2\pi \varrho^n}\int_0^{2\pi} 2U(\varsigma) e^{-in\phi} d\phi \\ &= iV(0) + \frac{1}{2\pi}\int_0^{2\pi} U(\varsigma) d\phi + \frac{1}{2\pi}\int_0^{2\pi} U(\varsigma)\left(\sum_{n=1}^{2\pi} 2\frac{z^n}{\varsigma^n}\right) d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} U(\varsigma)\left[1 + 2\sum_{n=1}^{\infty} \frac{z^n}{\varsigma^n}\right] d\phi + iV(0) \\ &= \frac{1}{2\pi}\int_0^{2\pi} U(\varsigma)\left[\frac{\varsigma + z}{\varsigma - z}\right] d\phi + iV(0), \end{aligned}$$

which yields equation (1.8.4). If $p_0 = 1$, then $iV(0) = 0$. $\square$

1.9. Scope of dissertation

The present work is devoted to the study of certain subclasses of univalent analytic functions defined in the unit disk $\Delta = \{z : |z| < 1\}$.

In Chapter 2, we extend S to the class consisting of p-valent functions of the form

$$\mathcal{A}(p,m) := \left\{ f(z) : f(z) = z^p + \sum_{m}^{\infty} a_n z^n \right\},$$

$p, m \in \mathcal{N} = \{1, 2, \ldots\}$. Note that $S \subset \mathcal{A} = \mathcal{A}(1,1)$.

We define a subclass $T_g[p,m,\alpha]$ of functions in $\mathcal{A}(p,m)$ with negative coefficients and obtain coefficient inequalities. Distortion and growth estimates for functions in this class as well as inclusion and closure properties are also determined. A representation theorem is derived and the Bernardi integral operator is studied.

Let Σ_p be the class of meromorphic functions of the form $f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k$ defined in the unit disk Δ. Functions in Σ_p are analytic in the punctured unit disk $\Delta^* = \Delta - \{0\}$. In Chapter 3, inequalities are obtained for meromorphic functions in Σ_p which are associated with the Liu-Srivastava linear operator $H_p^{l,m}$ and the multiplier transform $I_p(n, \lambda)$. In addition, we obtain sufficient conditions for $f \in \Sigma_p$ to satisfy a growth inequality.

CHAPTER 2

A CLASS OF P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

2.1. Introduction

Consider the class of p-valent analytic functions

$$\mathcal{A}(p,m)=\left\{f(z)=z^p+\sum_{n=m}^{\infty}a_nz^n,\quad (p,m\in\mathcal{N},z\in\Delta)\right\}.$$

Note that $S\subset\mathcal{A}=\mathcal{A}(1,1)$.

Let $f,g\in\mathcal{A}(p,m)$. The convolution of $f(z)=z^p+\sum_{n=m}^{\infty}a_nz^n$ and $g(z)=z^p+\sum_{n=m}^{\infty}b_nz^n$, denoted by $(f*g)(z)$, is defined by

$$(f*g)(z)=z^p+\sum_{n=m}^{\infty}a_nb_nz^n.$$

For a fixed function $g(z)=z^p+\sum_{n=m}^{\infty}g_nz^n$, the subclass $T_g(p,m,\alpha)$ is defined by

$$T_g(p,m,\alpha)=\left\{f\in\mathcal{A}(p,m):\Re\left(\frac{(f*g)(z)}{z^p}\right)>\alpha,\quad 0\le\alpha<1\right\}.$$

If now $g(z)$ has positive coefficients, i.e.,

$$g(z)=z^p+\sum_{n=m}^{\infty}g_nz^n,\quad (g_n>0),$$

then we shall denote the subclass of $T_g(p,m,\alpha)$ consisting of functions

$$f(z)=z^p-\sum_{n=m}^{\infty}a_nz^n,\quad (a_n\ge 0)$$

by $T_g[p,m,\alpha]$. We shall prove several interesting properties of functions with negative coefficients in the class $T_g[p,m,\alpha]$.

The class of TH which is considered by Janteng et.al [**13**] is a special case of $T_g[p, m, \alpha]$. In fact, $TH = T_g[1, 2, \alpha]$ where

$$g(z) = \frac{z}{(1-z)^2} + \sum_{n=2}^{\infty} \beta n(n-1) z^n$$
$$= z + \sum_{n=2}^{\infty} n[1 + \beta(n-1)] z^n.$$

2.2. Coefficient inequalities

THEOREM 2.2.1. *If* $f \in \mathcal{A}(p, m)$ *satisfies* $\sum_{n=m}^{\infty} |a_n g_n| \leq 1 - \alpha$,$0 \leq \alpha < 1$, *then* $f \in T_g(p, m, \alpha)$.

PROOF. Let $\sum_{n=m}^{\infty} |a_n g_n| \leq 1 - \alpha$. Then

$$\left|\frac{(f*g)(z)}{z^p} - 1\right| = \left|\frac{z^p + \sum_{n=m}^{\infty} a_n g_n z^n}{z^p} - 1\right|$$
$$\leq \sum_{n=m}^{\infty} |a_n g_n||z|^{n-p} \leq \sum_{n=m}^{\infty} |a_n g_n|$$
$$\leq 1 - \alpha.$$

Thus $\left|\frac{(f*g)(z)}{z^p} - 1\right| \leq 1 - \alpha$. Since $-\Re w \leq |w|$, we have

$$-\Re\left(\frac{(f*g)(z)}{z^p} - 1\right) \leq 1 - \alpha,$$

or

$$\Re\frac{(f*g)(z)}{z^p} \geq \alpha.$$

Thus $f \in T_g(p, m, \alpha)$ □

THEOREM 2.2.2. *Let* $f(z) = z^p - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0$. *The function* f *belongs to* $T_g[p, m, \alpha]$ *if and only if*

$$\sum_{n=m}^{\infty} a_n g_n \leq 1 - \alpha.$$

Proof. Let $f \in T_g[p,m,\alpha]$. By letting $z \to 1^-$ through real values in the condition $\Re\left\{\frac{(f*g)(z)}{z^p}\right\} > \alpha$, we get $1 - \sum_{n=m}^{\infty} a_n g_n \geq \alpha$. The converse follows from Theorem 2.2.1. $\square$

Corollary 2.2.3. *If $f \in T_g[p,m,\alpha]$, then for $n \geq m$,*

$$a_n \leq \frac{1-\alpha}{g_n},$$

and the result is sharp for

$$f(z) = z^p - \frac{1-\alpha}{g_n} z^n.$$

Proof. Since $f \in T_g[p,m,\alpha]$, we have

$$a_n g_n \leq \sum_{n=m}^{\infty} a_n g_n \leq 1 - \alpha$$

or

$$a_n \leq \frac{1-\alpha}{g_n}.$$

Clearly equality holds for

$$f(z) = z^p - \frac{1-\alpha}{g_n} z^n.$$

$\square$

2.3. Growth and distortion inequalities

Corollary 2.3.1. *If $f \in T_g[p,m,\alpha]$, then*

$$r^p - \frac{1-\alpha}{g_m} r^m \leq |f(z)| \leq r^p + \frac{1-\alpha}{g_m} r^m, \quad |z| = r < 1,$$

provided $g_n \geq g_m \geq 1$. These inequalities are sharp for

$$f(z) = z^p - \frac{1-\alpha}{g_m} z^m.$$

PROOF. For $f(z) \in T_g[p, m, \alpha]$, we have

$$g_m \sum_{n=m}^{\infty} a_n \leq \sum_{n=m}^{\infty} g_n a_n \leq 1 - \alpha,$$

or

$$\sum_{n=m}^{\infty} a_n \leq \frac{1-\alpha}{g_m}.$$

Therefore for $|z| = r$,

$$\begin{aligned} |f(z)| &\leq |z|^p + |z|^m \sum_{n=m}^{\infty} |z|^{n-m} a_n \\ &\leq |z|^p + |z|^m \sum_{n=m}^{\infty} a_n \\ &\leq r^p + r^m \frac{1-\alpha}{g_m}. \end{aligned}$$

To verify the sharpness, consider the function $f(z) = z^p - \frac{1-\alpha}{g_m} z^n$. For this function, we have

$$|f(z)| = |z^p - \frac{1-\alpha}{g_m} z^m| = |z|^p |1 - \frac{1-\alpha}{g_m} z^{m-p}|.$$

We next choose z on $|z| = r$ such that $z = (-1)^{\frac{1}{m-p}} r = e^{\frac{i\pi(1+2k)}{m-p}} r, k = 0, 1 \ldots, m-p-1$. The other inequality is shown in a similar manner. $\square$

COROLLARY 2.3.2. *If $f \in T_g[p, m, \alpha]$, then for $|z| \leq r$*

$$pr^{p-1} - r^{m-1} m \frac{1-\alpha}{g_m} \leq |f'(z)| \leq pr^{p-1} + \frac{m(1-\alpha)}{g_m} r^{m-1},$$

provided $\frac{g_n}{n}$ is increasing function of n. The result is sharp for the function $f(z) = z^p - \frac{1-\alpha}{g_m} z^m$.

PROOF. Since $\frac{g_n}{n}$ is increasing, we have

$$\sum_{n=m}^{\infty} n a_n \leq \frac{m}{g_m} \sum_{n=m}^{\infty} a_n g_n \leq \frac{m}{g_m}(1-\alpha),$$

and hence

$$\begin{aligned}|f'(z)| &\leq p|z|^{p-1} + \sum_{n=m}^{\infty} n|z|^{n-1}a_n \\ &\leq pr^{p-1} + r^{m-1}\frac{m}{g_m}(1-\alpha).\end{aligned}$$

For the function $f(z) = z^p - \frac{1-\alpha}{g_m}z^n$, we have

$$\begin{aligned}|f'(z)| &= |pz^{p-1} - \frac{m(1-\alpha)}{g_m}z^{m-1}| \\ &= |z|^{p-1}|p - \frac{m(1-\alpha)}{g_m}z^{m-p}|\end{aligned}$$

Choose z on $|z| = r$ so that $z = (-1)^{\frac{1}{m-p}}r = e^{\frac{i\pi(1+2k)}{m-p}}r, \quad k = 0, 1\ldots, m-p-1$. This shows that the result is sharp.

The other inequality is shown in a similar manner. □

2.4. Inclusion and closure theorems

THEOREM 2.4.1. *Let* $g_n^* \leq g_n, \quad \alpha_2 \leq \alpha_1$. *Then*

$$T_g[p, m, \alpha_1] \subseteq T_{g^*}[p, m, \alpha_2].$$

PROOF. If $f \in T_g[p, m, \alpha_1]$, then $\sum_{n=m}^{\infty} a_n g_n \leq 1 - \alpha_1$. Now

$$\sum_{n=m}^{\infty} a_n g_n^* \leq \sum_{n=m}^{\infty} a_n g_n \leq 1 - \alpha_1 \leq 1 - \alpha_2$$

so that $f \in T_g[p, m, \alpha_2]$. □

THEOREM 2.4.2. *Let* $f_i \in T_g[p, m, \alpha]$ *for* $i = 1, 2, \ldots, N$ *where* $0 \leq \alpha < 1$ *and* $\sum_{i=1}^{N} \beta_i = 1$. *Then*

$$f(z) = \sum_{i=1}^{N} \beta_i f_i(z) \in T_g[p, m, \alpha].$$

Proof. Let $f_i \in T_g[p, m, \alpha]$, be given by $f_i(z) = z^p + \sum_{n=m}^{\infty} a_{n,i} z^n$.

Then

$$\sum_{n=m}^{\infty} a_{n,i} g_n \leq 1 - \alpha$$

and therefore

$$\begin{aligned}
\sum_{n=m}^{\infty} \sum_{i=1}^{N} \beta_i a_{n,i} g_n &= \sum_{i=1}^{N} \beta_i \left(\sum_{n=m}^{\infty} a_{n,i} g_n \right) \\
&\leq (1 - \alpha) \sum_{i=1}^{N} \beta_i \\
&= 1 - \alpha.
\end{aligned}$$

This proves the result. $\square$

Theorem 2.4.3. *If $h(z) = z^p + \sum_{n=m}^{\infty} h_n z^n$ satisfies $0 \leq h_n \leq 1$, and $f \in T_g[p, m, \alpha]$, then $f * h \in T_g[p, m, \alpha]$.*

Proof. For $f \in T_g[p, m, \alpha]$, we have

$$\sum_{n=m}^{\infty} a_n g_n h_n \leq \sum_{n=m}^{\infty} a_n g_n \leq 1 - \alpha.$$

Therefore $f * h \in T_g[p, m, \alpha]$. $\square$

If $f \in \mathcal{A}(p, m)$, the Bernardi integral operator is the function

$$F(z) = \frac{c + p}{z^c} \int_0^z t^{c-1} f(t) dt, \quad (c > -p).$$

Theorem 2.4.4. *If $f(z) \in T_g[p, m, \alpha]$, then the Bernardi integral*

$$F(z) \in T_g[p, m, \alpha], \quad (c > -p).$$

Proof. Since

$$
\begin{aligned}
F(z) &= \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \\
&= \frac{c+p}{z^c}\int_0^z \left(t^{p+c-1} - \sum_{n=m}^{\infty} a_n t^{c-1+n}\right) dt \\
&= \frac{c+p}{z^c}\left[\frac{t^{p+c}}{p+c} - \sum_{n=m}^{\infty}\frac{a_n t^{c+n}}{c+n}\right]_0^z \\
&= \frac{c+p}{z^c}\left[\frac{z^{p+c}}{p+c} - \sum_{n=m}^{\infty}\frac{a_n z^{c+n}}{c+n}\right],
\end{aligned}
$$

we have

$$F(z) = z^p - \sum_{n=m}^{\infty}\frac{c+p}{c+n}a_n z^n.$$

Thus $F(z) = f(z) * h(z)$ where

$$h(z) = z^p + \sum_{n=m}^{\infty}\frac{c+p}{c+n}z^n = z^p + \sum_{n=m}^{\infty} h_n z^n$$

with $h_n = \frac{c+p}{c+n}$. Since $0 \le h_n \le 1$, the result follows from Theorem 2.4.3. □

2.5. A representation theorem

Theorem 2.5.1. *Define the function*

$$
\begin{aligned}
&h_p(z) = z^p \\
&h_n(z) = z^p - \tfrac{1-\alpha}{g_n} z^n, \quad \textit{for} \quad n = m, m+1, \ldots
\end{aligned}
$$

Let $\lambda_n \ge 0$ *and* $\lambda_p + \sum_{n=m}^{\infty}\lambda_n = 1$. *Then* $f \in T_g[p, m, \alpha]$ *if and only if*

$$f(z) = \lambda_p h_p(z) + \sum_{n=m}^{\infty}\lambda_n h_n(z). \tag{2.5.1}$$

Proof. If f is given by equation (2.5.1), then

$$f(z) = \lambda_p z^p + \sum_{n=m}^{\infty} \lambda_n \left[z^p - \frac{1-\alpha}{g_n} z^n \right]$$
$$= z^p - \sum_{n=m}^{\infty} \lambda_n \frac{1-\alpha}{g_n} z^n.$$

Now

$$\sum_{n=m}^{\infty} \lambda_n \frac{1-\alpha}{g_n} g_n = \sum_{n=m}^{\infty} \lambda_n (1-\alpha)$$
$$= (1-\alpha)(1-\lambda_p)$$
$$\leq 1-\alpha.$$

Thus from Theorem 2.2.2 , $f \in T_g[p,m,\alpha]$. Conversely, let $f \in T_g[p,m,\alpha]$. Define λ_n by

$$\lambda_n = \frac{g_n a_n}{1-\alpha}$$

and

$$\lambda_p = 1 - \sum_{n=m}^{\infty} \lambda_n$$

Thus

$$f(z) = z^p - \sum_{n=m}^{\infty} a_n z^n = z^p - \sum_{n=m}^{\infty} \lambda_n \frac{1-\alpha}{g_n} z^n$$
$$= z^p + \sum_{n=m}^{\infty} \lambda_n (h_n - z^p)$$
$$= (1 - \sum_{n=m}^{\infty} \lambda_n) z^p + \sum \lambda_n h_n$$
$$= \lambda_p z^p + \sum_{n=m}^{\infty} \lambda_n h_n.$$

This completes the proof. $\square$

CHAPTER 3

INEQUALITIES FOR MEROMORPHIC FUNCTIONS DEFINED BY CERTAIN LINEAR OPERATORS

3.1. Introduction

Let Σ_p denote the class of all analytic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (z \in \Delta^* := \{z \in \mathcal{C} : 0 < |z| < 1\}) \tag{3.1.1}$$

and let $\Sigma_1 := \Sigma$. For two functions $f(z)$ given by (3.1.1) and $g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{3.1.2}$$

For $\alpha_j \in \mathcal{C}$ $(j = 1, 2, \ldots, l)$ and $\beta_j \in \mathcal{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m + 1; l, m \in \mathcal{N}_0 := \{0, 1, 2, \ldots\})$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathcal{N} := \{1, 2, 3 \ldots\}). \end{cases}$$

Corresponding to the function

$$h_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := z^{-p} \; {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z),$$

the Liu-Srivastava operator [**18**] $H_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)$ is defined by the Hadamard product

$$\begin{aligned} H_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) f(z) &= h_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * f(z) \\ &= \tfrac{1}{z^p} + \textstyle\sum_{n=1-p}^{\infty} \tfrac{(\alpha_1)_{n+p} \dots (\alpha_l)_{n+p}}{(\beta_1)_{n+p} \dots (\beta_m)_{n+p}} \tfrac{a_n z^n}{(n+p)!}. \end{aligned}$$

To make the notation simple, we write

$$H_p^{l,m}[\alpha_1] f(z) := H_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) f(z).$$

Special cases of the Liu-Srivastava linear operator includes the meromorphic analogue of the Carlson-Shaffer linear operator $\mathcal{L}_p(a, c) := H_p^{(2,1)}(1, a; c)$ considered by Liu [**16**] and the operator $D^{n+1} := \mathcal{L}_p(n + p, 1)$ investigated by Yang [**26**] (which is analogous to the Ruscheweyh derivative operator), and the operator $J_{c,p} = \mathcal{L}_p(c, c + 1)$ studied, for example, by Uralegaddi and Somanatha [**25**]. Note that

$$J_{c,p} f = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) dt \quad (c > 0).$$

Motivated by the operator studied by Aouf and Hossen [**3**] (see also [**6, 16, 22**]), we define the operator $I_p(n, \lambda)$ on Σ_p by the following infinite series

$$I_p(n, \lambda) f(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{k + \lambda}{\lambda - p} \right)^n a_k z^k \quad (\lambda \geq p). \tag{3.1.3}$$

In our present investigation, we extend the following two theorems of Miller and Mocanu [**20**] for functions associated with the Liu-Srivastava linear operator $H_p^{l,m}$ and the multiplier transform $I_p(n,\lambda)$.

Definition 3.1.1. Let $H = H(\Delta)$ denote the class of functions analytic in Δ. For n a positive integer and $a \in \mathcal{C}$, let

$$H[a,n] = \{f \in H : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\},$$

with $H_0 = H[0,1]$.

Definition 3.1.2. We denote by Q the set of functions q that are analytic and injective on $\overline{\Delta} \setminus E(q)$, where

$$E(q) = \{\varsigma \in \partial\Delta; \lim_{z\to\varsigma} q(z) = \infty\}$$

and are such that $q'(\varsigma) \neq 0$ for $\varsigma \in \partial\Delta \setminus E(q)$.

Definition 3.1.3. Let Ω be a set in $\mathcal{C}, q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\Psi : \mathcal{C}^3 \times \Delta \to \mathcal{C}$ that satisfy the admissibility condition $\Psi(r,s,t;z) \notin \Omega$ whenever $r = q(\varsigma), s = m\varsigma q'(\varsigma)$,

$$\Re\left\{\frac{t}{s} + 1\right\} \geq m\Re\left\{\frac{\varsigma q''(\varsigma)}{q'(\varsigma} + 1\right\},$$

$z \in \Delta, \varsigma \in \partial\Delta \setminus E(q)$ and $m \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

Let $\Delta_M = \{w : |w| < M\}$. The function $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, satisfies $q(\Delta) = \Delta_M, \quad q(0) = 1, E(q) = \emptyset$ and $q \in Q$.

We set $\Psi_n[\Omega, M, a] := \Psi_n[\Omega, q]$ and in the special case when $\Omega = \Delta$, we denote the class by $\Psi_n[M, a]$.

THEOREM 3.1.4. [**20**, Theorem 2.3h, p.34] *Let $p \in H[a, n]$*

(1) *If $\psi \in \psi_n[\Omega, M, a]$, then*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \Rightarrow |p(z)| < M$$

(2) *If $\psi \in \psi_n[M, a]$, then*

$$|\psi(p(z), zp'(z), z^2p''(z); z)| < M \Rightarrow |p(z)| < M$$

To prove our results, we need the following lemma due to Miller and Mocanu.

LEMMA 3.1.5. [**20**, cf. Lemma 2.2a, p. 19; Lemma 2.2e, p. 25] *Let $w(z) = a + w_m z^m + \cdots$ be analytic in Δ with $w(z) \not\equiv a$ and $m \geq 1$. If $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$, then $z_0 w'(z_0) = kw(z_0)$ and $\Re\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq k$, where k is real and $k \geq m$.*

3.2. Inequalities associated with the Liu-Srivastava linear operator

We begin with the following definition for a class of functions.

DEFINITION 3.2.1. Let G_1 be the set of complex-valued functions $g(r, s, t) : \mathcal{C}^3 \to \mathcal{C}$ such that

(1) $g(r, s, t)$ is continuous in a domain $D \subset \mathcal{C}^3$,

(2) $(0,0,0) \in D$ and $|g(0,0,0)| < 1$,

(3) $\left|g\left(e^{i\theta}, \frac{k+\alpha_1-1}{\alpha_1}e^{i\theta}, \frac{L+\alpha_1(2k+\alpha_1-1)e^{i\theta}}{\alpha_1(\alpha_1+1)}\right)\right| \geq 1$,

whenever $\left(e^{i\theta}, \frac{k+\alpha_1-1}{\alpha_1}e^{i\theta}, \frac{L+\alpha_1(2k+\alpha_1-1)e^{i\theta}}{\alpha_1(\alpha_1+1)}\right) \geq 1$ for all $\theta, L, k \geq 1$ satisfying $\Re(e^{-i\theta}L) \geq k(k-1)$.

Making use of Lemma 3.1.5, we first prove the theorem below.

THEOREM 3.2.2. *Let $g(r,s,t) \in G_1$. If $f(z) \in \Sigma_p$ satisfies*

$\left(z^{p+1}H_p^{l,m}[\alpha_1]f(z), z^{p+1}H_p^{l,m}[\alpha_1+1]f(z), z^{p+1}H_p^{l,m}[\alpha_1+2]f(z)\right) \in D \subset \mathcal{C}^3$

and for $z \in \Delta$,

$\left|g\left(z^{p+1}H_p^{l,m}[\alpha_1]f(z), z^{p+1}H_p^{l,m}[\alpha_1+1]f(z), z^{p+1}H_p^{l,m}[\alpha_1+2]f(z)\right)\right| < 1$,

then we have

$$\left|z^{p+1}H_p^{l,m}[\alpha_1]f(z)\right| < 1, \qquad (z \in \Delta).$$

PROOF. Define $w(z)$ by

$$w(z) := z^{p+1}H_p^{l,m}[\alpha_1]f(z). \tag{3.2.1}$$

Then

$$\begin{aligned} w(z) &= z^{p+1}\left(\frac{1}{z^p} + \sum_{n=1-p}^{\infty} \frac{(\alpha_1)_{n+p}\ldots(\alpha_l)_{n+p}}{(\beta_1)_{n+p}\ldots(\beta_m)_{n+p}} \frac{a_n z^n}{(n+p)!}\right) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}\ldots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\ldots(\beta_m)_{n-1}} \frac{a_{n-p-1}z^n}{(n-1)!} \end{aligned}$$

is analytic in Δ and $w(z) \neq 0$ at least for one $z \in \Delta$. By differentiating (3.2.1) and then multiplying by z, we get

$$zw'(z) = (p+1)z^{p+1}H_p^{l,m}[\alpha_1]f(z) + z^{p+1}z[H_p^{l,m}[\alpha_1]f(z)]'.$$

From the relation

$$(3.2.2)\quad \alpha_1 H_p^{l,m}[\alpha_1+1]f(z) = z[H_p^{l,m}[\alpha_1]f(z)]' + (\alpha_1+p)H_p^{l,m}[\alpha_1]f(z),$$

we get

$$(3.2.3)\qquad \alpha_1 z^{p+1} H_p^{l,m}[\alpha_1+1]f(z) = zw'(z) + (\alpha_1-1)w(z).$$

Differentiating (3.2.3) and multiplying by z yields

$$\alpha_1 z^{p+1} z[H_p^{l,m}[\alpha_1+1]f(z)]' + \alpha_1(p+1)z^{p+1}H_p^{l,m}[\alpha_1+1]f(z)$$
$$= z^2w''(z) + \alpha_1 zw'(z).$$

Using (3.2.2) in the above equation, we get

$$\alpha_1 z^{p+1}[(\alpha_1+1)H_p^{l,m}[\alpha_1+2]f(z) - (\alpha_1+p+1)H_p^{l,m}[\alpha_1+1]f(z)]$$
$$+ \alpha_1(p+1)z^{p+1}H_p^{l,m}[\alpha_1+1]f(z)$$
$$= z^2w''(z) + \alpha_1 zw'(z).$$

Using (3.2.3) in the above equation, we get

$$\alpha_1(\alpha_1+1)z^{p+1}H_p^{l,m}[\alpha_1+2]f(z) = z^2w''(z) + 2\alpha_1 zw'(z) + \alpha_1(\alpha_1-1)w(z).$$

If $|z^{p+1}H_p^{l,m}[\alpha_1]f(z)| < 1$ is false, then there exists z_0 with $|z_0| = r_0 < 1$ such that

$$|w(z_0)| = \max_{|z|\le|z_0|} |w(z)| = 1.$$

Letting $w(z_0) = e^{i\theta}$ and using Lemma 3.1.5, we see that

$$\begin{aligned}
z_0^{p+1} H_p^{l,m}[\alpha_1] f(z_0) &= e^{i\theta}, \\
z_0^{p+1} H_p^{l,m}[\alpha_1 + 1] f(z_0) &= \frac{z_0 w'(z_0) + (\alpha_1 - 1) w(z_0)}{\alpha_1} \\
&= \frac{k w(z_0) + (\alpha_1 - 1) w(z_0)}{\alpha_1} \\
&= \frac{k + \alpha_1 - 1}{\alpha_1} e^{i\theta},
\end{aligned}$$

and

$$\begin{aligned}
z_0^{p+1} H_p^{l,m}[\alpha_1 + 2] f(z_0) &= \frac{z_0^2 w''(z_0) + 2\alpha_1 z_0 w'(z_0) + \alpha_1(\alpha_1 - 1) w(z_0)}{\alpha_1(\alpha_1 + 1)} \\
&= \frac{L + 2\alpha_1 k w(z_0) + \alpha_1(\alpha_1 - 1) w(z_0)}{\alpha_1(\alpha_1 + 1)} \\
&= \frac{L + \alpha_1(2k + \alpha_1 - 1) e^{i\theta}}{\alpha_1(\alpha_1 + 1)},
\end{aligned}$$

where $L = z_0^2 w''(z_0)$ and $k \geq 1$. Further, by an application of Lemma 3.1.5, we have

$$\Re\left\{ \frac{z_0 w''(z_0)}{w'(z_0)} \right\} = \Re\left\{ \frac{z_0^2 w''(z_0)}{k e^{i\theta}} \right\} \geq k - 1,$$

or $\Re\{e^{-i\theta} L\} \geq k(k-1)$. Since $g(r, s, t) \in G_1$, we have

$$\begin{aligned}
&\left| g\left(z_0^{p+1} H_p^{l,m}[\alpha_1] f(z_0), z_0^{p+1} H_p^{l,m}[\alpha_1 + 1] f(z_0), z_0^{p+1} H_p^{l,m}[\alpha_1 + 2] f(z_0) \right) \right| \\
&= \left| g\left(e^{i\theta}, \frac{k + \alpha_1 - 1}{\alpha_1} e^{i\theta}, \frac{L + \alpha_1(2k + \alpha_1 - 1) e^{i\theta}}{\alpha_1(\alpha_1 + 1)} \right) \right| \geq 1,
\end{aligned}$$

which contradicts the hypothesis of Theorem 3.2.2. Therefore we conclude that

$$|w(z)| = \left| z^{p+1} H_p^{l,m}[\alpha_1] f(z) \right| < 1 \quad (z \in \Delta).$$

This completes the proof of Theorem 3.2.2. $\square$

Corollary 3.2.3. *If $f(z) \in \Sigma_p$ satisfies*

$$|z^{p+1} H_p^{l,m}[\alpha_1 + 1] f(z)| < 1 \quad (\Re\alpha_1 \geq 0)$$

then

$$|z^{p+1} H_p^{l,m}[\alpha_1] f(z)| < 1.$$

Proof. The result follows by taking $g(r,s,t) : \mathcal{C}^3 \to \mathcal{C}$ to be the function defined by $g(r,s,t) = s$. For this function,

$$\left| g\left(e^{i\theta}, \frac{k+\alpha_1-1}{\alpha_1}e^{i\theta}, \frac{L+\alpha_1(2k+\alpha_1-1)e^{i\theta}}{\alpha_1(\alpha_1+1)}\right)\right| = \left|\frac{k+\alpha_1-1}{\alpha_1}e^{i\theta}\right| \geq 1$$

if

$$|k+\alpha_1-1| \geq |\alpha_1|$$

or if

$$(k-1)^2 + 2(k-1)\Re\alpha_1 + |\alpha_1|^2 \geq |\alpha_1|^2$$

or if

$$2\Re\alpha_1 \geq -\frac{(k-1)^2}{k-1} = 1-k.$$

Since $k \geq 1$, this condition holds if $\Re\alpha_1 \geq 0$. Thus $g \in G_1$. The result follows from Theorem 3.2.2. □

Definition 3.2.4. Let G_2 be the set of complex-valued functions $g(r,s,t) : \mathcal{C}^3 \to \mathcal{C}$ such that

(1) $g(r,s,t)$ is continuous in a domain $D \subset \mathcal{C}^3$,

(2) $(1,1,1) \in D$ and $|g(1,1,1)| < 1$,

(3) $\left|g\left(e^{i\theta}, \frac{1+k+\alpha_1 e^{i\theta}}{1+\alpha_1}, \frac{2+k+\alpha_1 e^{i\theta}}{\alpha_1+2} + \frac{k[1-k+L+\alpha_1 e^{i\theta}]}{(\alpha_1+2)(1+k+\alpha_1 e^{i\theta})}\right)\right| \geq 1,$

whenever $\left(e^{i\theta}, \frac{1+k+\alpha_1 e^{i\theta}}{1+\alpha_1}, \frac{2+k+\alpha_1 e^{i\theta}}{\alpha_1+2} + \frac{k[1-k+L+\alpha_1 e^{i\theta}]}{(\alpha_1+2)(1+k+\alpha_1 e^{i\theta})}\right) \in D$ with $\Re(L) \geq k-1$ for real θ, $\alpha_1 \in \mathcal{C}$ and $k \geq 1$.

Making use of the above Lemma 3.1.5, we now prove

Theorem 3.2.5. *Let $g(r,s,t) \in G_2$. If $f(z) \in \Sigma_p$ satisfies*

$$\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}\right) \in D \subset \mathcal{C}^3$$

and

$$\left|g\left(\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}, \frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)}\right)\right| < 1, \quad (z \in \Delta),$$

then we have

$$\left|\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\right| < 1, \qquad (z \in \Delta).$$

Proof. Define $w(z)$ by

$$w(z) = \frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \tag{3.2.4}$$

$$\begin{aligned} w(z) &= \frac{\frac{1}{z^p} + \sum_{n=1-p}^{\infty} \frac{(\alpha_1+1)_{n+p}\dots(\alpha_l+1)_{n+p}}{(\beta_1+1)_{n+p}\dots(\beta_m+1)_{n+p}} \frac{a_n z^n}{(n+p)!}}{\frac{1}{z^p} + \sum_{n=1-p}^{\infty} \frac{(\alpha_1)_{n+p}\dots(\alpha_l)_{n+p}}{(\beta_1)_{n+p}\dots(\beta_m)_{n+p}} \frac{a_n z^n}{(n+p)!}} \\ &= 1 + d_1 z + d_2 z^2 + \dots \end{aligned}$$

Then $w(z)$ is analytic in Δ and $w(z) \neq 1$ at least for one $z \in \Delta$. By logarithmic differentiation yields

$$\frac{zw'(z)}{w(z)} = \frac{z[H_p^{l,m}[\alpha_1+1]f(z)]'}{H_p^{l,m}[\alpha_1+1]f(z)} - \frac{z[H_p^{l,m}[\alpha_1]f(z)]'}{H_p^{l,m}[\alpha_1]f(z)},$$

and from relation (3.2.2), we get

$$
\begin{aligned}
\frac{zw'(z)}{w(z)} &= \frac{(\alpha_1+1)H_p^{l,m}[\alpha_1+2]f(z)-(\alpha_1+p+1)H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)} \\
&- \frac{\alpha_1[H_p^{l,m}[\alpha_1+1]f(z)-(\alpha_1+p)H_p^{l,m}[\alpha_1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} \\
&= \frac{(\alpha_1+1)H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)} - (\alpha_1+1+p) \\
&- \alpha_1 w(z) + (\alpha_1+p).
\end{aligned}
$$

Thus

$$
\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)} = \frac{1}{1+\alpha_1}\left[1+\alpha_1 w(z)+\frac{zw'(z)}{w(z)}\right].
$$

Differentiating logarithmically, we get

$$
\begin{aligned}
&\frac{z[H_p^{l,m}[\alpha_1+2]f(z)]'}{H_p^{l,m}[\alpha_1+2]f(z)} - \frac{z[H_p^{l,m}[\alpha_1+1]f(z)]'}{H_p^{l,m}[\alpha_1+1]f(z)} \\
&= \frac{\frac{z^2w''(z)}{w(z)}+\alpha_1 zw'(z)+\frac{zw'(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2}{1+\alpha_1 w(z)+\frac{zw'(z)}{w(z)}}.
\end{aligned}
$$

From the relation (3.2.2) we get

$$
\begin{aligned}
&\frac{(\alpha_1+2)H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)} - (\alpha_1+2+p) \\
&- (1+\alpha_1 w(z)+\frac{zw'(z)}{w(z)}) + (\alpha_1+1+p) \\
&= \frac{\frac{z^2w''(z)}{w(z)}+\alpha_1 zw'(z)+\frac{zw'(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2}{1+\alpha_1 w(z)+\frac{zw'(z)}{w(z)}}
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{H_p^{l,m}[\alpha_1+3]f(z)}{H_p^{l,m}[\alpha_1+2]f(z)} \\
&= \frac{1}{2+\alpha_1}\left[2+\alpha_1 w(z)+\frac{zw'(z)}{w(z)}+\frac{\alpha_1 zw'(z)+\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2}{1+\alpha_1 w(z)+\frac{zw'(z)}{w(z)}}\right]
\end{aligned}
$$

If $\left|\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\right| < 1$ is false, then there exists z_0 with $|z_0| = r_0 < 1$ such that

$$|w(z_0)| = \max_{|z|\le|z_0|} |w(z)| = 1.$$

Letting $w(z_0) = e^{i\theta}$ and using Lemma 3.1.5, we see that

$$\frac{H_p^{l,m}[\alpha_1+1]f(z_0)}{H_p^{l,m}[\alpha_1]f(z_0)} = e^{i\theta},$$
$$\frac{H_p^{l,m}[\alpha_1+2]f(z_0)}{H_p^{l,m}[\alpha_1+1]f(z_0)} = \frac{1+k+\alpha_1 e^{i\theta}}{1+\alpha_1},$$

and

$$\frac{H_p^{l,m}[\alpha_1+3]f(z_0)}{H_p^{l,m}[\alpha_1+2]f(z_0)} = \frac{1}{2+\alpha_1}\left[2+k+\alpha_1 e^{i\theta} + \frac{k(1-k)+kL+k\alpha_1 e^{i\theta}}{k+1+\alpha_1 e^{i\theta}}\right],$$

where $L = \frac{z_0^2 w''(z_0)}{kw(z_0)}$ satisfies $\Re(L) \ge k-1$, $k \ge 1$. Since $g(r,s,t) \in G_2$, we have

$$\left|g\left(\frac{H_p^{l,m}[\alpha_1+1]f(z_0)}{H_p^{l,m}[\alpha_1]f(z_0)}, \frac{H_p^{l,m}[\alpha_1+2]f(z_0)}{H_p^{l,m}[\alpha_1+1]f(z_0)}, \frac{H_p^{l,m}[\alpha_1+3]f(z_0)}{H_p^{l,m}[\alpha_1+2]f(z_0)}\right)\right|$$
$$= \left|g\left(e^{i\theta}, \frac{1+k+\alpha_1 e^{i\theta}}{1+\alpha_1}, \frac{1}{\alpha_1+2}\left[2+k+\alpha_1 e^{i\theta} + \frac{k[1-k+L+\alpha_1 e^{i\theta}]}{k+1+\alpha_1 e^{i\theta}}\right]\right)\right| \ge 1,$$

which contradicts the hypothesis of Theorem 3.2.5. Therefore we conclude that

$$|w(z)| = \left|\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\right| < 1 \quad (z \in \Delta).$$

This completes the proof of Theorem 3.2.5. $\square$

3.3. Inequalities associated with multiplier transform

In this section, we prove results similar to Theorem 3.2.2 and Theorem 3.2.5 for functions defined by multiplier transform. We need the following:

Definition 3.3.1. Let G_3 be the set of complex-valued functions $g(r,s,t) : \mathcal{C}^3 \to \mathcal{C}$ such that

(1) $g(r,s,t)$ is continuous in a domain $D \subset \mathcal{C}^3$,

(2) $(0,0,0) \in D$ and $|g(0,0,0)| < 1$,

(3) $\left|g\left(e^{i\theta}, \frac{\lambda-p+k-1}{\lambda-p}e^{i\theta}, \frac{L+[(2\lambda-2p-1)k+(\lambda-p-1)^2]e^{i\theta}}{(\lambda-p)^2}\right)\right| \geq 1,$

whenever $\left(e^{i\theta}, \frac{\lambda-p+k-1}{\lambda-p}e^{i\theta}, \frac{L+[(2\lambda-2p-1)k+(\lambda-p-1)^2]e^{i\theta}}{(\lambda-p)^2}\right) \in D$, with $\Re(e^{-i\theta}L) \geq k(k-1)$ for real $\theta, \lambda \geq 0$ and $k \geq 1$.

Theorem 3.3.2. *Let $g(r,s,t) \in G_3$. If $f(z) \in \Sigma_p$ satisfies*

$$\big(z^{p+1}I_p(n,\lambda)f(z), z^{p+1}I_p(n+1,\lambda)f(z), z^{p+1}I_p(n+2,\lambda)f(z)\big) \in D \subset \mathcal{C}^3$$

and

$$\big|g\big(z^{p+1}I_p(n,\lambda)f(z), z^{p+1}I_p(n+1,\lambda)f(z), z^{p+1}I_p(n+2,\lambda)f(z)\big)\big| < 1, \quad (z \in \Delta),$$

then we have

$$|z^{p+1}I_p(n,\lambda)f(z)| < 1, \qquad (z \in \Delta).$$

Proof. Define $w(z)$ by

$$w(z) := z^{p+1}I_p(n,\lambda)f(z). \tag{3.3.1}$$

$$= z + \sum_{k=1-p}^{\infty}\left(\frac{k+\lambda}{\lambda-p}\right)^n a_k z^{k+p+1}$$

$$= z + \sum_{k=2}^{\infty}\left(\frac{k-p-1}{\lambda-p}\right)^n a_{k-p-1} z^k$$

Then $w(z)$ is analytic in Δ and $w(z) \neq 0$ at least for one $z \in \Delta$. Differentiating (3.3.1) and multiplying by z yields

$$zw'(z) = z^{p+1}z(I_p(n,\lambda)f(z))' + (p+1)z^{p+1}I_p(n,\lambda)f(z).$$

From the relation

$$(\lambda - p) I_p(n+1,\lambda) f(z) = z[I_p(n,\lambda) f(z)]' + \lambda I_p(n,\lambda) f(z) \tag{3.3.2}$$

we get

$$zw'(z) = z^{p+1}[(\lambda - p) I_p(n+1,\lambda) f(z) - \lambda I_p(n,\lambda) f(z)] + (p+1) w(z).$$

Then we get

$$zw'(z) = (\lambda - p) z^{p+1} I_p(n+1,\lambda) f(z) - \lambda w(z) + (p+1) w(z).$$

And hence

$$(\lambda - p) z^{p+1} I_p(n+1,\lambda) f(z) = zw'(z) + (\lambda - p - 1) w(z). \tag{3.3.3}$$

Differentiating (3.3.3) and multiplying by z yields

$$\begin{aligned} z_0^{p+1} I_p(n,\lambda) f(z_0) &= e^{i\theta}, \\ z_0^{p+1} I_p(n+1,\lambda) f(z_0) &= \frac{kw(z_0) + (\lambda - p - 1) w(z_0)}{\lambda - p} \\ &= \frac{(k + \lambda - p - 1) e^{i\theta}}{\lambda - p} \end{aligned}$$

and

$$\begin{aligned} z_0^{p+1} I_p(n+2,\lambda) f(z_0) &= \frac{L + (2\lambda - 2p - 1) k w(z) + (\lambda - p - 1)^2 w(z)}{(\lambda - p)^2} \\ &= \frac{L + [(2\lambda - 2p - 1) k + (\lambda - p - 1)^2] e^{i\theta}}{(\lambda - p)^2}, \end{aligned}$$

where $L = z_0^2 w''(z_0)$ and $k \geq 1$. Further, an application of Lemma 3.1.5 we obtain that

$$\Re\left\{ \frac{z_0 w''(z_0)}{w'(z_0)} \right\} = \Re\left\{ \frac{z_0^2 w''(z_0)}{k e^{i\theta}} \right\} \geq k - 1,$$

or $\Re\{e^{-i\theta}L\} \geq k(k-1)$. Since $g(r,s,t) \in G_3$, we have

$$\left|g\big(z^{p+1}I_p(n,\lambda)f(z_0), z^{p+1}I_p(n+1,\lambda)f(z_0), z^{p+1}I_p(n+2,\lambda)f(z_0)\big)\right|$$
$$= \left|g\left(e^{i\theta}, \frac{\lambda-p+k-1}{\lambda-p}e^{i\theta}, \frac{L+[(2\lambda-2p-1)k+(\lambda-p-1)^2]e^{i\theta}}{(\lambda-p)^2}\right)\right| \geq 1,$$

which contradicts the hypothesis of Theorem 3.3.2. Therefore we conclude that

$$|w(z)| = |z^{p+1}I_p(n,\lambda)f(z)| < 1, \quad (z \in \Delta).$$

This completes the assertion of Theorem 3.3.2. $\square$

COROLLARY 3.3.3. *If $f(z) \in \Sigma_p$ satisfies*

$$|z^{p+1}I_p(n,\lambda+1)f(z)| < 1, \quad (\Re\lambda > p)$$

then

$$|z^{p+1}I_p(n,\lambda)f(z)| < 1.$$

PROOF. The result follows by defining $g(r,s,t): \mathcal{C}^3 \to \mathcal{C}$ by $g(r,s,t) = s$. For this function, we have

$$\left|g\left(e^{i\theta}, \frac{\lambda-p+k-1}{\lambda-p}e^{i\theta}, \frac{L+[(2\lambda-2p-1)k+(\lambda-p-1)^2]e^{i\theta}}{(\lambda-p)^2}\right)\right| = \left|\frac{\lambda-p+k-1}{\lambda-p}e^{i\theta}\right| \geq 1$$

Provided

$$(k-1)^2 + 2(k-1)\Re(\lambda-p) + |\lambda-p|^2 \geq |\lambda-p|^2.$$

This is equivalent to

$$2\Re(\lambda-p) \geq -\frac{(k-1)^2}{k-1} = 1-k, \quad k \geq 1,$$

which holds if $\Re(\lambda-p) \geq 0$. Thus $g \in G_3$. $\square$

Definition 3.3.4. Let G_4 be the set of complex-valued functions $g(r,s,t) : \mathcal{C}^3 \to \mathcal{C}$ such that

(1) $g(r,s,t)$ is continuous in a domain $D \subset \mathcal{C}^3$,

(2) $(1,1,1) \in D$ and $|g(1,1,1)| < 1$,

(3) $\left| g\left(e^{i\theta}, e^{i\theta} + \frac{k}{\lambda-p}, + \frac{k\left[e^{i\theta} + \frac{L+1-k}{\lambda-p}\right]}{(\lambda-p)e^{i\theta}+k} \right) \right| \geq 1,$

whenever $\left(e^{i\theta}, e^{i\theta} + \frac{k}{\lambda-p}, e^{i\theta} + \frac{k}{\lambda-p} + \frac{k\left[e^{i\theta} + \frac{L+1-k}{\lambda-p}\right]}{(\lambda-p)e^{i\theta}+k} \right) \in D$ with $\Re L \geq k - 1$ for real θ, $\lambda \geq 0$ and real $k \geq 1$.

Theorem 3.3.5. *Let $g(r,s,t) \in G_4$. If $f(z) \in \Sigma_p$ satisfies*

$$\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)} \right) \in D \subset \mathcal{C}^3$$

and

$$\left| g\left(\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}, \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}, \frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)} \right) \right| < 1, \quad (z \in \Delta),$$

then we have

$$\left| \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \right| < 1, \qquad (z \in \Delta).$$

Proof. Define $w(z)$ by

$$\begin{aligned} w(z) &:= \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \\ &= \frac{z + \sum_{k=1-p}^{\infty} \left(\frac{k+\lambda}{\lambda-p}\right)^{n+1} a_k z^{k+p+1}}{z + \sum_{k=1-p}^{\infty} \left(\frac{k+\lambda}{\lambda-p}\right)^{n} a_k z^{k+p+1}} \quad = 1 + e_1 z + e_2 z^2 + \dots . \end{aligned}$$

Then $w(z)$ is analytic in Δ and $w(z) \neq 1$ at least for one $z \in \Delta$. By logarithmic differentiation, we get

$$\frac{zw'(z)}{w(z)} = \frac{z(I_p(n+1,\lambda)f(z))'}{I_p(n+1,\lambda)f(z)} - \frac{z(I_p(n,\lambda)f(z))'}{I_p(n,\lambda)f(z)}$$

By making use of (3.3.2) we get

$$\frac{zw'(z)}{w(z)} = \frac{(\lambda-p)I_p(n+2,\lambda)f(z)-(\lambda)I_p(n+1,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}$$
$$-\frac{(\lambda-p)I_p(n+1,\lambda)f(z)-\lambda I_p(n,\lambda)f(z)}{I_p(n,\lambda)f(z)}$$
$$= \frac{(\lambda-p)I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - (\lambda-p)w(z).$$

Then

$$\frac{(\lambda-p)I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} = \frac{zw'(z)}{w(z)} + (\lambda-p)w(z).$$

Differentiating logarithmically the above equation yields

$$\frac{z(I_p(n+2,\lambda)f(z))'}{I_p(n+2,\lambda)f(z)} - \frac{z(I_p(n+1,\lambda)f(z))'}{I_p(n+1,\lambda)f(z)}$$
$$= \frac{(\lambda-p)zw'(z) + \frac{z^2w''(z)}{w(z)} + \frac{zw'(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2}{(\lambda-p)w(z) + \frac{zw'(z)}{w(z)}}.$$

From the relation (3.3.2), we get

$$\frac{(\lambda-p)I_p(n+3,\lambda)f(z)-\lambda I_p(n+2,\lambda)f(z)}{I_p(n+2,\lambda)f(z)} - \frac{(\lambda-p)I_p(n+2,\lambda)f(z)-\lambda I_p(n+1,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}$$
$$= \frac{(\lambda-p)zw'(z) + \frac{z^2w''(z)}{w(z)} + \frac{zw'(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2}{(\lambda-p)w(z) + \frac{zw'(z)}{w(z)}}.$$

Then

$$\frac{(\lambda-p)I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)} - (\lambda-p)w(z) - \frac{zw'(z)}{w(z)}$$
$$= \frac{(\lambda-p)zw'(z) + \frac{z^2w''(z)}{w(z)} + \frac{zw'(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2}{(\lambda-p)w(z) + \frac{zw'(z)}{w(z)}}$$

and hence

$$\frac{I_p(n+3,\lambda)f(z)}{I_p(n+2,\lambda)f(z)}$$
$$= w(z) + \frac{1}{\lambda-p}\frac{zw'(z)}{w(z)} + \frac{1}{\lambda-p}\frac{\left[(\lambda-p)zw'(z) + \frac{z^2w''(z)}{w(z)} + \frac{zw'(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2\right]}{(\lambda-p)w(z) + \frac{zw'(z)}{w(z)}}.$$

If $\left|\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right| < 1$ is false, then there is exists z_0 with $|z_0| = r_0 < 1$ such that

$$|w(z_0)| = \max_{|z|\leq|z_0|} |w(z)| = 1.$$

Letting $w(z_0) = e^{i\theta}$ and using Lemma 3.1.5, we see that

$$\frac{I_p(n+1,\lambda)f(z_0)}{I_p(n,\lambda)f(z_0)} = e^{i\theta},$$
$$\frac{I_p(n+2,\lambda)f(z_0)}{I_p(n+1,\lambda)f(z_0)} = e^{i\theta} + \frac{k}{\lambda-p},$$

and

$$\frac{I_p(n+3,\lambda)f(z_0)}{I_p(n+2,\lambda)f(z_0)} = e^{i\theta} + \frac{k}{\lambda-p} + \frac{k\left[e^{i\theta} + \frac{L+1-k}{\lambda-p}\right]}{(\lambda-p)e^{i\theta}+k},$$

where $L = z_0w''(z_0)/w'(z_0)$ and $k \geq 1$. Further, an application of Lemma 3.1.5, we obtain that

$$\Re\left\{\frac{z_0w''(z_0)}{w'(z_0)}\right\} \geq k-1,$$

or $\Re L \geq k-1$. Since $g(r,s,t) \in G_4$, we have

$$\left|g\left(\frac{I_p(n+1,\lambda)f(z_0)}{I_p(n,\lambda)f(z_0)}, \frac{I_p(n+2,\lambda)f(z_0)}{I_p(n+1,\lambda)f(z_0)}, \frac{I_p(n+3,\lambda)f(z_0)}{I_p(n+2,\lambda)f(z_0)}\right)\right|$$
$$= \left|g\left(e^{i\theta}, e^{i\theta} + \frac{k}{\lambda-p}, e^{i\theta} + \frac{k}{\lambda-p} + + \frac{k\left[e^{i\theta} + \frac{L+1-k}{\lambda-p}\right]}{(\lambda-p)e^{i\theta}+k}\right)\right| \geq 1,$$

which contradicts the hypothesis of Theorem 3.3.2. Therefore we conclude that

$$|w(z)| = \left|\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right| < 1,$$

for all $z \in \Delta$. This completes the assertion of Theorem 3.3.5. $\square$

CHAPTER 4

SUMMARY

The present work is devoted to the study of certain subclasses of univalent analytic functions defined in the unit disk $\Delta = \{z : |z| < 1\}$.

In Chapter 2, we extend S to the class consisting of p-valent analytic functions

$$\mathcal{A}(p,m) := \left\{ f(z) : f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n \quad \text{is analytic in } \Delta \right\},$$

$p, m \in \mathcal{N} = \{1, 2, \ldots\}$ Note that $S \subset \mathcal{A} = \mathcal{A}(1,1)$.

We denote a subclass $T_g[p, m, \alpha]$ in $\mathcal{A}(p,m)$ with negative coefficients and obtain coefficient inequalities. Distortion and growth estimates for functions in this class as well as inclusion and closure properties are also determined. A representation theorem is derived and the Bernardi integral operator is studied.

Let Σ_p be the class of meromorphic functions of the form $f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k$ defined in the unit disk Δ. Functions in Σ_p are analytic in the punctured unit disk $\Delta^* = \Delta - \{0\}$. In Chapter 3, inequalities are obtained for meromorphic functions in Σ_p which are associated with the Liu-Srivastava linear operator $H_p^{l,m}$ and the multiplier transform $I_p(n, \lambda)$. In addition, we obtain sufficient conditions for $f \in \Sigma_p$ to satisfy a growth inequality.

REFERENCES

[1] R. Aghalary, R. M. Ali, S. B. Joshi and V. Ravichandran, Inequalities for analytic functions defined by certain linear operators, preprint.

[2] L. V. Ahlfors(1978), *Complex analysis*, Third edition, McGraw-Hill, New York.

[3] M. K. Aouf and H. M. Hossen(1993), New criteria for meromorphic p-valent starlike functions, *Tsukuba J. Math.* **17**(2) , 481–486.

[4] L. de Branges(1985),A proof of the Bierberbach conjecture, *Acta Math.*, **154** 137-152.

[5] N. E. Cho and H. M. Srivastava(2003), Argument estimates of certain analytic functions defined by a class of multiplier transformations, *Math. Comput. Modelling*, **37** 39–49.

[6] N. E. Cho and S. H. Lee(1996), Certain subclasses of meromorphic functions defined by subordination. II, Kyungpook Math. J. **36** (2), 283–291.

[7] N. E. Cho and T. H. Kim(2003), Multiplier transformations and strongly close-to-convex functions, *Bull. Korean Math. Soc.*, **40** , 399–410.

[8] N. E. Cho, O. S. Kwon and H. M. Srivastava(2004), Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations, J. Math. Anal. Appl. **300**(2), 505–520.

[9] J. Dziok, H.M. Srivastava(2003), Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transforms Spec. Funct.*, **14** 7-18.

[10] A. W. Goodman (1983), *Univalent functions. Vol. I*, Mariner, Tampa, FL.

[11] A. W. Goodman (1983), *Univalent functions. Vol. II*, Mariner, Tampa, FL.

[12] A. S. B. Holland(1980), *Complex function theory*, North-Holland, New York.

[13] A. Janteng, S. A. Halim and M. Darus, A new subclass of harmonic univalent functions, preprint.

[14] W. Kaplan(1966), *Introduction to analytic functions*, Addison-Wesley Publishing Co., Reading, Mass.

[15] E. Kreyszig(1999), *Advanced engineering mathematics*, Eighth edition, Wiley, New York.

[16] J.-L. Liu (2003), A linear operator and its applications on meromorphic p-valent functions, Bull. Inst. Math. Acad. Sinica **31**(1), 23–32.

[17] J.-L. Liu and S. Owa(1998), On certain meromorphic p-valent functions, *Taiwanese J. Math.*, **2**(1), 107–110.

[18] J.-L. Liu and H. M. Srivastava(2004), Classes of meromorphically multivalent functions associated with the generalized hypergeometric function, Math. Comput. Modelling **39**(1), 21–34.

[19] J.-L. Liu and H. M. Srivastava(2004), Subclasses of meromorphically multivalent functions associated with a certain linear operator, Math. Comput. Modelling **39**(1), 35–44.

[20] S. S. Miller and P. T. Mocanu(2000), Differential Subordinations: Theory and Applications. *Pure and Applied Mathematics No. 225*, Marcel Dekker, New York.

[21] S. Owa and H. M. Srivastava(1987), Univalent and starlike generalized bypergeometric functions, *Canad. J. Math.*, **39**, 1057–1077.

[22] J. Patel and P. Sahoo(2001), On certain subclasses of meromorphically p-valent functions, Bull. Calcutta Math. Soc. **93** (6), 455–464.

[23] H. Silverman (1975), Univalent functions with negative coefficients, Proc. Amer. Math. Soc. **51**, 109–116.

[24] H. M. Srivastava(2001), Some families of fractional derivative and other linear operators associated with analytic, univalent and multivalent functions, in: *Proc. International Conf. Analysis and its Applications*, K. S. Lakshmi, et. al (ed.), Allied Publishers Ltd, New Delhi, 209–243.

[25] B. A. Uralegaddi and C. Somanatha(1991), New criteria for meromorphic starlike univalent functions, Bull. Austral. Math. Soc. **43**(1), 137–140.

[26] D. Yang (1996), On a class of meromorphic starlike multivalent functions, Bull. Inst. Math. Acad. Sinica **24**(2), 151–157.